ACYLINDRICAL SURFACES IN 3-MANIFOLDS AND KNOT COMPLEMENTS

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Abstract. We consider closed acylindrical surfaces in 3-manifolds and in knot and link complements, and show that the genus of these surfaces is bounded linearly by the number of tetrahedra in a triangulation of the manifold and by the number of rational (or alternating) tangles in a projection of a link (or knot). For each g we find knots with tunnel number 2 and manifolds of Heegaard genus 3 containing acylindrical surfaces of genus g. Finally, we construct 3-bridge knots containing quasi-Fuchsian surfaces of unbounded genus, and use them to find manifolds of Heegaard genus 2 and homology spheres of Heegaard genus 3 containing infinitely many incompressible surfaces.

1. Introduction

A closed incompressible surface F embedded in a 3-manifold M is called *acylindrical* if the manifold $M_F = M - intN(F)$, obtained by cutting M along F contains no essential annuli (a properly embedded annulus in a 3-manifold is essential if it is incompressible and not boundary parallel). Acylindrical surfaces are interesting in connection with geometry, as every totally geodesic surface in a hyperbolic 3-manifold is acylindrical, and every acylindrical surface in a hyperbolic link complement is quasi-Fuchsian. Moreover, if F is an acylindrical surface in a closed, irreducible and atoroidal 3-manifold M then M_F admits a hyperbolic metric with totally geodesic boundary [21].

In [12] Hass proved that for the finite volume hyperbolic 3-manifolds there is a constant C, independent of the manifold, so that each acylindrical surface in a manifold M has genus at most $C \cdot vol(M)$. He used this result to show that in any compact 3-manifold there is only a finite number of acylindrical surfaces. It seems natural to ask if there are similar bounds which hold for all 3-manifolds and depend not on volume, but on some topological measures of complexity. Some candidates could be the number of tetrahedra in a triangulation or the Heegaard genus of the manifold, and in the case of knots and links, the crossing number, the bridge number or the tunnel number. Such bounds must exist in the case of the number of tetrahedra in a triangulation or the crossing number of a link, as there are only finitely many manifolds and links for each number n. We find explicit bounds in these cases, and furthermore show that there is a linear bound in terms of the number of rational tangles in a link projection or the number of alternating tangles in a prime knot projection.

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The fact that 3-manifolds with Heegaard genus 2 and the complements of knots with tunnel number 1 contain no separating acylindrical surfaces ([19], [4]) could suggest that -at least for small Heegaard genus or tunnel number- there could be bounds for the genus of such surfaces. We show here that for each g, there are tunnel number 2 knots which contain a closed acylindrical surface of genus g. By performing suitable Dehn surgeries, we get closed manifolds of Heegaard genus 3 which contain closed acylindrical surfaces of genus g. These examples show that Heegaard genus 3 manifolds and tunnel number 2 knots are already quite complicated.

We also consider what happens when the acylindrical assumption is weakened to require that there are no essential annuli running from the surface to a boundary torus (in the case of hyperbolic knots and links this means that the surface is quasi-Fuchsian). We show that a knot that can be decomposed into two alternating tangles cannot contain any quasi-Fuchsian surfaces in its complement. On the other hand, we find hyperbolic 3-bridge knots whose complements contain infinitely many quasi-Fuchsian surfaces. These knots have an essential branched surface which carries quasi-Fuchsian surfaces of arbitrarily high genus. These examples show that there are no bounds for the genus of quasi-Fuchsian surfaces based on volume, crossing number or the number of tetrahedra. Finally, by means of suitable Dehn fillings and double covers, we produce manifolds of Heegaard genus 2 and homology spheres of Heegaard genus 3 which contain infinitely many incompressible surfaces. These examples are interesting, for it seems that all known examples of hyperbolic manifolds with infinitely many surfaces have noncyclic homology, and in the case of knots with infinitely many surfaces, it seems that the only known explicit examples are some satellite knots (see for example [17]). The examples are also interesting for the study of surfaces in the complement of 3-bridge knots, as they supplement results of Finkelstein and Moriah [6], who showed that many 3-bridge knots contain an incompressible but meridionally compressible surface, and of Ichihara and Ozawa [15], who proved that any closed surface in the complement of a 3-bridge knot is meridionally compressible or annular.

2. Bounds for the genus of acylindrical surfaces

Proposition 1. *If a closed 3-manifold M admits a (pseudo)triangulation with n tetrahedra then the genus of a 2-sided closed acylindrical surface in M is at most $\frac{n+1}{2}$.*

Proof. Let T be a (pseudo)triangulation of M with n tetrahedra, and denote by T_i the i-skeleton of T.

Let F be an incompressible surface in M in normal position with respect to the triangulation, so F intersects the faces of the tetrahedra along arcs and the interior of the tetrahedra along discs which are triangles or squares. Assume further that F has been isotoped to minimize the number of intersections with T_1. Let $\overline{F}$ be the boundary of a regular neighborhood N of F. As F is two-sided, $\overline{F}$ consists of two copies of F. By definition F is acylindrical iff $M - intN$ contains no essential annuli.

The edges of $\overline{F}$ in each face of a tetrahedron split the face into triangles, quadrangles, pentagons and/or hexagons, and each edge is adjacent to a quadrangle (which lies in N). Call an edge *good* if the other adjacent region (which lies in $M - intN$) is also a quadrangle. Notice that if an embedded curve c in $\overline{F}$ is made

of good edges, then the union of these adjacent quadrangles in $M - intN$ forms an annulus A that joins c with another curve c' in $\overline{F}$.

We claim that if c is essential in $\overline{F}$ then the annulus A is essential. Otherwise A would be isotopic to an annulus A' bounded by c and c' in $\overline{F}$ (in particular, c must be 2-sided in $\overline{F}$). As F is 2-sided in M, then A' is parallel to an annulus A'' in F and the isotopy from A' to A can be used to isotope A'' (pushing it even further across A) to reduce the number of intersections of F with T_1.

So if $\overline{F}$ is acylindrical, then the good edges of $\overline{F}$ carry no embedded essential curves, and so they carry no essential curves at all. But as the edges of $\overline{F}$ split $\overline{F}$ into discs, they must carry all of $H_1(\overline{F})$.

So there must be at least as many non-good edges in $\overline{F}$ as the rank of $H_1(\overline{F})$. As the number of non-good edges in a face of a tetrahedron is at most 6, the total number of non-good edges in $\overline{F}$ is at most $12n$, so $12n \geq rank\ H_1(\overline{F}) = 2 \cdot genus$ $\overline{F}$, and so the genus of F is at most $3n$.

In order to get the better estimate one needs to look more carefully at the graph Q formed by the edges of $\overline{F}$. Divide the non-good edges in each tetrahedron in two classes: those lying in triangles of $\overline{F}$ that cut off outermost corners of the tetrahedron will be called *fair edges* and the others (which may lie in squares or triangles) will be called *bad edges* (see Figure 1).

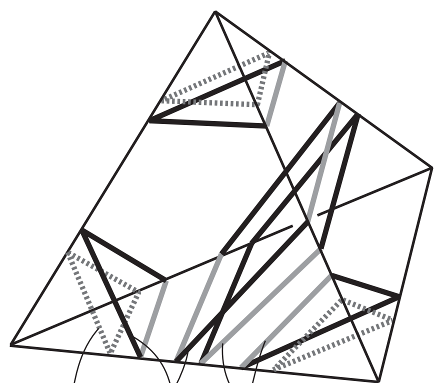

Figure 1

Let Q_g and Q_f denote the subgraphs of Q made of good edges and fair edges respectively.

Observe that Q_g and Q_f are disjoint, that is, have no vertices in common. As the components of Q_f lie in the links of the vertices of a triangulation of the manifold M, all curves contained in Q_f are contractible in M, and so as $\overline{F}$ is incompressible then Q_f contains only trivial curves of $\overline{F}$. All curves contained in Q_g are also trivial because $\overline{F}$ is acylindrical. So as $Q_g \cup Q_f$ contains only trivial curves of $\overline{F}$, by attaching to $Q_g \cup Q_f$ some of the complementary pieces of $\overline{F}$ we obtain a (possibly empty or disconnected) simply connected subcomplex $\overline{F}_S$ of $\overline{F}$.

Now the Euler characteristic of $\overline{F}$ is $\chi(\overline{F}) = \chi(\overline{F}_S) + v - e + f$ where v, e, and f count the vertices, edges and faces of $\overline{F}$ that do not lie in $\overline{F}_S$. So e counts some bad edges -some others may lie in $\overline{F}_S$- and f counts the triangles and squares adjacent to them. It can be shown directly that in each tetrahedron Δ, every subcollection

f_Δ of the set of squares and triangles of $\overline{F} \cap \Delta$ with bad edges satisfies the inequality $\frac{1}{2}e_\Delta - f_\Delta \leq 2$. In particular, we may take f_Δ to be the set of squares and triangles with bad edges not contained in $\overline{F}_S$. As $\overline{F}$ has two components and each of them contains a component of $\overline{F}_S$ or a vertex, it follows that

$\chi(\overline{F}) \geq 2 - e + f = 2 + \sum_{\Delta \in T_3} -\frac{1}{2}e_\Delta + f_\Delta \geq 2 - 2n$

and so $genus(F) = \frac{1}{4} rank H_1(\overline{F}) = \frac{1}{4}(4 - \chi(\overline{F})) \leq \frac{1}{4}(2 + 2n)$. □

The genus of an acylindrical surface in a manifold is not bounded in terms of its Heegaard genus, as we show in Section 3. However, there is a bound depending on the complexity of a Heegaard splitting. Let $M = H \cup H'$ be a Heegaard splitting of M of genus g, and let $D_1, D_2, ..., D_g$ and $D'_1, D'_2, ..., D'_g$ be discs splitting H and H' into 3-balls B and B'. The complexity of the Heegaard splitting with respect to these discs is just the minimal intersection number between the boundaries of the discs. The complexity of a Heegaard splitting is the minimum complexity among all such systems of discs.

Proposition 2. *If a closed 3-manifold M admits an irreducible Heegaard splitting of genus g and complexity n then the genus of a closed acylindrical surface in M is at most $(n - \frac{3}{2}g)$.*

Proof. Let $M = H \cup H'$ be a Heegaard splitting of M of genus g as above, with $\cup D_i$ meeting $\cup D'_j$ in n points. Let F be an acylindrical surface in M. As F is incompressible, we may assume that F meets H' along g stacks of parallel discs in $N(D'_j)$ (some stacks may be empty). We may also assume that F meets B along discs and that it meets each D_i along stacks of parallel arcs connecting different components of $\partial D_i \cap N(D'_j)$.

As before, consider the graph of intersection Q of $\overline{F}$ with $\partial H \cup_i D_i$. Call an edge of Q on D_i *good* if it is an interior arc of a stack, otherwise call it *bad*. Call an edge of Q in ∂H *good* if it is part of the boundary of an interior disc of a stack. Otherwise (i.e., if it is part of the boundary of an outermost disc of a stack) call it *fair*. See Figure 2.

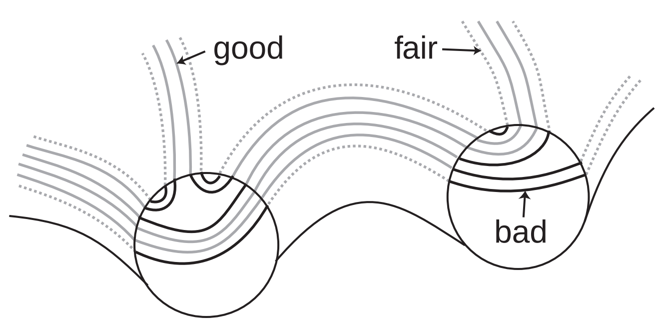

Figure 2

Observe that the subgraphs Q_g and Q_f made of good edges and fair edges do not meet. As the components of Q_f are contained in the boundaries of discs in H' then Q_f carries no essential curves of $\overline{F}$, and as F is acylindrical we may assume as in the proof of 2.1 that Q_g carries no essential curves either.

So as $Q_g \cup Q_f$ carries no essential curves and Q splits $\overline{F}$ into discs, the rank of $H_1(\overline{F})$ is bounded above by the number of bad edges. If D_i meets the D'_j in n_i

points then $n_i > 1$ (because the Heegaard splitting is irreducible) and D_i contains at most $4n_i - 6$ bad edges, so the rank of $H_1(\overline{F})$ is at most $\sum_{D_i}(4n_i - 6) = 4n - 6g$ and so the genus of F is at most $n - \frac{3}{2}g$. □

There are other ways of measuring the complexity of a Heegaard splitting, for example, by means of the curve complex, as defined in [14]. Note however that no such bound for the genus of acylindrical surfaces exists for this complexity, for in fact, all the examples constructed in Section 3 have a Heegaard splitting of genus 3 which comes from a certain bridge presentation of a knot, and then by a similar proof to theorem 1.4 of [14], the distance in the curvecomplex is ≤ 2.

We now consider bounds for the genus of acylindrical surfaces in the exterior of knots and links in the 3-sphere.

Proposition 3. *If k is a knot or link with n crossings then the genus of a closed acylindrical surface in the exterior of k is at most $\frac{3}{2}n - 3$.*

Proof. Draw k on a projection sphere S, except for the crossings which lie on the surface of n small spheres S_1, S_2,...,S_n. Let S_0 be the part of the projection sphere outside the S_i's. Then $S_0 \cup_i S_i$ cuts S^3 into $n+2$ polyhedral balls B^-, B^+ and B_1, B_2, ..., B_n, with faces determined by the equators of the bubbles and the arcs of k. If F is an incompressible surface in the exterior of k then F can be isotoped to meet B^+ and B^- along discs, meet each B_i along parallel saddle-shaped discs, and meet their faces along arcs. See Figure 3.

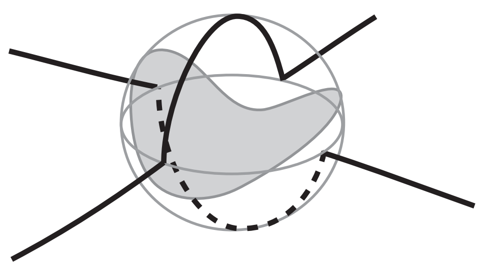

Figure 3

Let $\overline{F}$ be the boundary of a regular neighborhood of F, and let Q be the graph of intersection of $\overline{F}$ with $S_0 \cup_i S_i$. So Q splits $\overline{F}$ into discs. As before, consider the edges of Q on each face of $S_0 \cup_i S_i$, call those that have parallel edges on both sides *good*, those which are closest to arcs of k and are parallel to them *fair*, those lying on some S_i and parallel to an arc of ∂S_0 which contain a point of k are also fair, and all the others are called *bad* edges (so the faces of the B_i's contain no bad edges). See Figure 4.

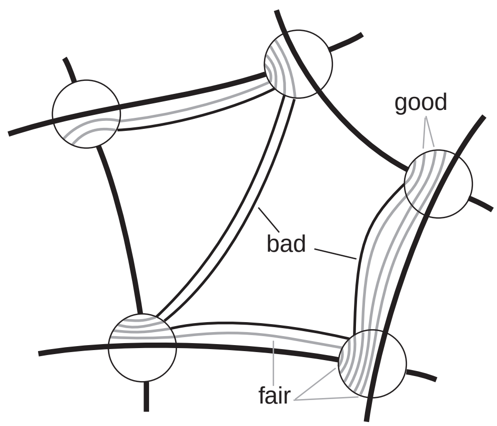

Figure 4

Again the subgraphs Q_g and Q_f of Q are disjoint, and if $\overline{F}$ is acylindrical then Q_g carries no essential curves of $\overline{F}$. On the other hand, Q_f can be regarded as lying on the boundary tori of a regular neighborhood of the link k. But as $\overline{F}$ is acylindrical, there can be no essential annuli running from $\overline{F}$ to k, so Q_f contains no essential curves of $\overline{F}$. So, as the graph Q carries all of $H_1(\overline{F})$, there must be at least as many bad edges as the rank of $H_1(\overline{F})$. As there are at most $3i-6$ bad edges on each i-gon determined by the projection of k into S, the number of bad edges in $\overline{F}$ is at most
$\sum_{i-\text{gons in } P}(3i-6) = 3(2(\text{ arcs of } k \text{ in } S)) - 6\,(\text{regions determined by } k \text{ in } S)$
$= 12n - 6(2+n) = 6n - 12$

So the rank of $H_1(\overline{F})$ is at most $6n-12$ and the genus of F is at most $\frac{6}{4}n - \frac{12}{4}$. $\square$

After we proved proposition 3, we learned that Agol and D. Thurston, following Lackenby [16], showed that the volume of a hyperbolic knot of link is bounded above by $10v_3(t(D)-1)$ where v_3 is the volume of a hyperbolic ideal tetrahedra and t is the twist number of k (the minimum number of twists in a diagram of k, where a twist is a string of 2-gons or a crossing in the diagram). Agol has also shown [1] that if a hyperbolic manifold M has an acylindrical surface of genus g, then $Vol(M) \geq 4v_3(g-1)$. It follows that the genus of an acylindrical surface in the exterior of a hyperbolic link k is at most $\frac{5}{2}t$. These results suggested the following.

Recall that a tangle is a 3-ball B together with two properly embedded arcs. The tangle is rational if the arcs are isotopic (rel ∂) to arcs in ∂B. We will say that a knot or link k in S^3 is decomposed into tangles if there is a sphere S and 3-balls B_1, B_2, ..., B_n each intersecting S in a disc, so that $k \cap B_i$ is a tangle, and the part of k outside these balls is a collection of arcs lying on $S_0 = S - int(\cap B_i)$.

Theorem 1. *If a link is decomposed into n rational tangles, then the genus of a closed acylindrical surface in its complement is at most $2n-4$.*

Proof. Draw the projection of the link k as the union of n rational tangles in the interior of n disjoint spheres S_1, S_2,...,S_n joined by $2n$ disjoint arcs in the projection sphere. Let S_0 be the part of the projection sphere outside these spheres. Then $S_0 \cup_i S_i$ cuts S^3 into polyhedral balls B_+, B_- and B_1,B_2,...,B_n with faces determined by the equators of the spheres and the arcs of k in S_0.

If F is an incompressible surface in the exterior of k we may isotope F so that intersects B^- and B^+ along discs, and intersects S and the hemispheres of each S_i along arcs. Moreover, as $k \cap B_i$ is a rational tangle, we may isotope F to intersect $B_i - k$ along parallel discs that separate the strings of the tangle, and we may assume that their boundaries meet each hemisphere of S_i along 2 or 3 families of parallel arcs -2 if the tangle is a crossing and 3 otherwise (a single family of parallel arcs implies that the discs are vertical and the tangle has no crossings of k).

Let $\overline{F}$ be the boundary of a regular neighborhood N of F. The intersection of $\overline{F}$ with $S_0 \cup_i S_i$ gives a cell decomposition of $\overline{F}$ and cuts the faces of $S_0 \cup_i S_i$ into quadrangles that lie in N and other polygons that lie in $S^3 - N$; as before, let Q be the graph of intersection. Call an edge of Q in a face of $S_0 \cup_i S_i$ *good* if the adjacent polygon in $S^3 - N$ is a quadrangle with another edge on $\overline{F}$ (so the two edges are parallel in that face). Otherwise, call an edge in Q *fair* if it is adjacent to a quadrangle in S_0 with a side in $k \cap S_0$ that is adjacent to another quadrangle in S_0 with a side in Q (so both edges of Q are parallel to this arc of k) or if it is adjacent to a polygon in a hemisphere of S_i with exactly 2 sides in Q (so the other sides lie in the equator and are separated by points of $k \cap S_i$). Call the other edges of Q *bad*. Note that edges lying on some S_i and parallel to an arc of ∂S_0 which contain a point of k are bad. See Figure 5.

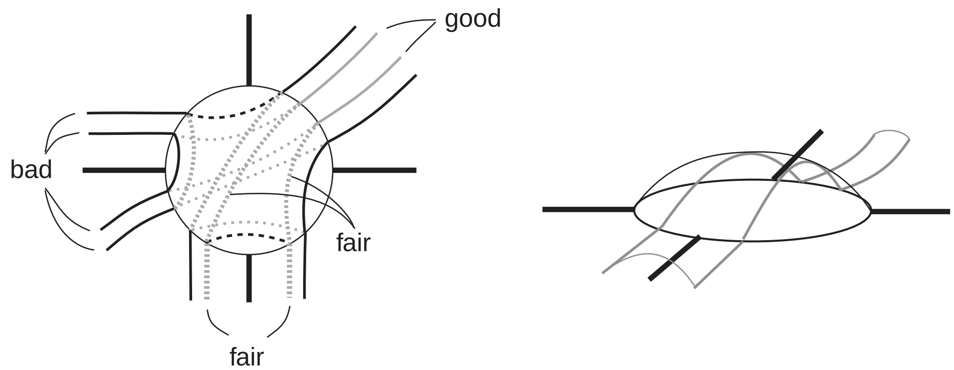

Figure 5

One can use the fair edges as well as the good edges to construct annuli for $\overline{F}$, by taking the quadrangles that lie between two fair edges in $S_0 \cup_i S_i$ (but that may intersect k) and pushing them outside the corresponding S_i, or if they lie in S_0, to the side of S_0 that doesn't contain an edge of Q connecting the two fair edges (there can't be connecting edges on both sides because the union of the four edges would be a meridian of k, and so F would be meridionally compressible). This creates quadrangles in $S^3 - k$ connecting pairs of fair edges, and one can see that the quadrangles corresponding to consecutive fair or good edges match well.

As before, if a simple essential curve in $\overline{F}$ is made of good and fair edges then the annulus formed by the union of the adjacent quadrangles is essential or else F could be isotoped to reduce its intersection with $S_0 \cup_i S_i$. So, if F is acylindrical, the subgraph of Q consisting of the good and fair edges cannot contain any essential curve of $\overline{F}$, so it is contained in a simply connected subcomplex $\overline{F}_S$ of $\overline{F}$. Again, as $\overline{F}$ has two components and Q divides them into discs,

$\chi(\overline{F}) = \chi(\overline{F}_S) + v - e + f \geq 2 - e + f$ where v, e, and f count the vertices, edges and discs in $\overline{F} - \overline{F}_S$, and so $rankH_1(\overline{F}) = (4 - \chi(\overline{F}) \leq 2 + e - f$.

As there are at most 4 bad edges and 8 fair edges on each S_i, all contained in the 2 outermost discs of $\overline{F} \cap B_i$, the number of bad edges minus the number of discs that contain them in $\cup S_i$ is at most $2n$.

There are at most $i-3$ families of parallel edges on each face of S_0 determined by $i > 1$ arcs of k, not including the families of edges parallel to the arcs of k, and they produce at most $2i - 6$ bad edges on each face. If an arc of k has parallel families edges of Q on both sides, then there are two bad edges in these families, for the edges closest to k are fair. If an arc of k has only edges of Q on one side, then there are two bad edges in this family, since in this case the edge closest to k is not fair.

So, if no face of S_0 is a monogon the number of bad edges in S_0 is at most $\sum_{edgesofk} 2 + \sum_{i-\text{gons in } P} (2i-6) = 4n + 8n - 6(2+n) = 6n - 12$.

When $i = 1$, the previous formula undercounts the number of bad edges in the monogon as -3 instead of 0 -there are no edges in the monogon as they could be isotoped into B_i to eliminate two intersection curves of $\overline{F}$ with S_i-. In this case there cannot be bad edges around the endpoints of the monogon in S_i and so the discs of intersection of $\overline{F}$ with S_i are vertical and the tangle is trivial -unless $\overline{F}$ does not meet S_i at all, so there is an overcount on the number of bad edges in $\cup S_i$ by at least 2 and also on the number of bad edges in the face of S_0 adjacent to the monogon. So the previous bound also holds when some faces of S_0 are monogons.

Finally observe that since $\overline{F}$ has 2 components and each of them must meet B_+ and B_-, there must be at least 2 discs of $\overline{F} - \overline{F}_S$ inside each of these balls.

So, $genus(F) = \frac{1}{4} rankH_1(\overline{F}) \leq \frac{1}{4}(2 + e - f) \leq \frac{1}{4}(2 + 2n + (6n - 12) - 4) = 2n - \frac{7}{2}$ □

Consider a tangle as above, i.e., it is determined by the intersection of a 3-ball B with a link k, so that $B \cap S$ is a disc, where S is a projection sphere, and $k \cap \partial B$ consists of 4 points lying on S. We say that the tangle is alternating if its arcs can be isotoped, keeping ∂B fixed, to have an alternating projection on the sphere S. Note that each rational tangle is alternating. The next result extends Theorem 1 to allow alternating tangles.

Theorem 2. *If a prime knot is decomposed into alternating tangles, n of them rational, then the genus of a closed acylindrical surface in its complement is at most $2n - 4$.*

The proof is based on the following:

Claim 1. *Let k be a nonseparable link or a knot and S a sphere that meets k in 4 points. Then each acylindrical surface F in $S^3 - k$ is isotopic to one that either i) is disjoint from S or ii) intersects S in one curve or iii) meets one of the components of $S^3 - k - S$ along parallel discs.*

Proof. The sphere S separates k into two tangles. Isotope F to minimize its intersection with the 4-punctured sphere $S - k$. The intersection then contains no trivial curves, and as F is meridionally incompressible then it does not contain curves surrounding only one puncture, so all the curves c_1,c_2,...c_n in which F intersects S must be parallel in $S - k$. As F is acylindrical, if there is more than one c_i then the annuli connecting two of them in S cannot be essential, so either one annulus is

isotopic (rel ∂) to an annulus in F (and the isotopy can be used to remove two c_i's) or all the c_i's bound discs of F. So at least one of them, say c_1, bounds a disc D_1 in F that lies completely on one side of S. But then, as all c_i's are parallel to ∂D_1, one can draw parallel discs D_i in $S^3 - k$ on that side of S that meet F at c_i (and nowhere else). The union of the discs bounded by the c_i's in F and the D_i's form spheres in $S^3 - k$, and if k is a knot or a nonseparable link these spheres bound balls in $S^3 - k$, so the D_i's must be isotopic to the discs in F, and the isotopy reduces the number of curves unless the discs in F were already on one side of S. □

Claim 2. *If k is a prime knot and $k \cap B_i$ is an alternating tangle, then every acylindrical surface in the complement of k can be isotoped to meet $B_i - k$ along parallel discs or be disjoint from it.*

Proof of theorem. Assume for the moment that claim 2 is true, and isotope the surface F to meet only the B_i's corresponding to separable tangles. To estimate the genus of F we would like to count the number of bad edges and discs of $\overline{F}$ that contain them by replacing each nonseparable tangle in the diagram of k by a trivial tangle to get a knot k' and counting the bad edges of $\overline{F}$ in its diagram.

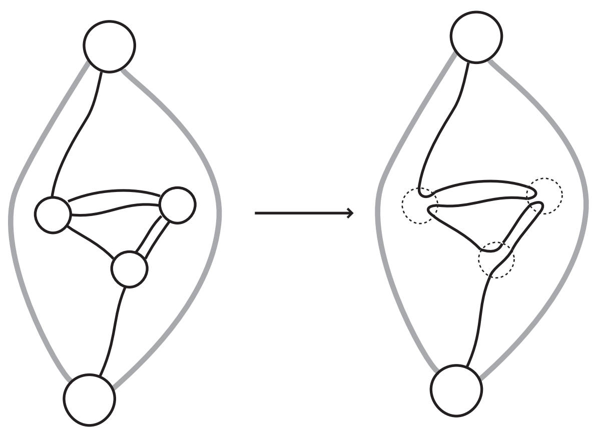

Figure 6

Now some bad edges in the diagram of k may become fair in the diagram of k' as in Figure 6, but in this case we may regard them as originally being "almost fair" - there is a quadrangle joining them that lies above or below the nonseparable tangles that were between them in the diagram of k. The quadrangles corresponding to almost fair edges match well with the other quadrangles corresponding to good and fair pairs of edges, so they can be used as well to construct annuli for $\overline{F}$. So the same bound for the number of bad edges and discs -and therefore the same bound for the genus of F- holds. □

The proof of claim 2 is based on the following extension of the Meridional Lemma of Menasco [18].

Lemma 1. *If a link k intersects a ball B in an alternating tangle, then every meridionally incompressible surface in the complement of k can be isotoped to intersect B along copies of a surface that separates the strings of the tangle.*

Proof. Draw B as a round ball with $k \cap B$ lying in an equatorial disc except at the crossings, that lie on the surface of small "bubbles" B_1, B_2,....as in Figure 3. Let ∂B_{i+} and ∂B_{i-} be the hemispheres of ∂B_i, and let D_0 denote the part of the equatorial disc outside the bubbles. Let $D_+ = D_0 \cup_i \partial B_{i+}$ and $D_- = D_0 \cup_i \partial B_{i-}$, and let B_+ and B_- be the parts of B above and below D_+ and D_-.

If F is a meridionally incompressible surface in the complement of k then by isotoping F to minimize its intersection with $\partial B \cup D \cup_i \partial B_i$ we can assume that F meets ∂B along parallel curves that separate 2 points of $\partial B \cap k$ from the other 2, that F meets D and each hemisphere of ∂B and ∂B_i along arcs and to meet B_+ and B_- along discs and each B_i along parallel saddle-shaped discs. So F intersects D_+ and D_- along curves and arcs with endpoints in ∂B.

Following Menasco, one can show that the curves and arcs of intersection of F with D_+ (and similarly with D_-) have the following properties:

1. As F is incompressible, each curve (and each arc) crosses at least one bubble.
2. As F is meridionally incompressible, each curve (or arc) crosses each bubble at most once.
3. As the diagram of $k \cap B$ is alternating, if a curve (or arc) crosses two bubbles B_i and B_j in succession, then the 2 arcs $k \cap \partial B_{i+}$ and $k \cap \partial B_{j+}$ lie on opposite sides of the curve. See Figure 7.

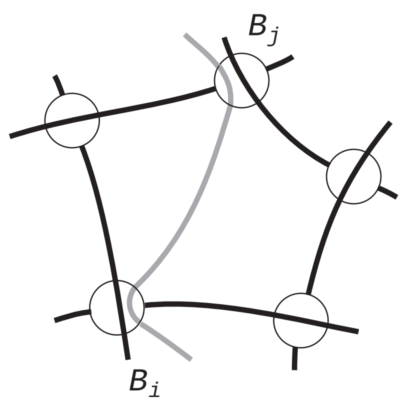

Figure 7

So there can be no closed curves in D_+, because by properties 1 and 3 an innermost such curve would have to leave an arc of $k \cap \partial B_{i+}$ inside (so there would be another curve inside) unless the curve crossed the same bubble twice, contradicting property 2.

Let k^1 and k^2, k_1 and k_2 be the 4 segments of $k \cap D_0$ that start on ∂D_0, and end in overcrossings or undercrossings of k respectively. Note that ∂D_0 encounters them in the order k^1, k_1, k^2, k_2, for otherwise there is an arc on D_0 separating the strings of the tangle, but then the knot will be composite. Properties 1, 2 and 3 for arcs imply that each outermost arc in D_+ goes around k_1 or k_2 and so every arc in D_+ must separate k_1 from k_2. See Figure 8a.

Now let F_0 be a surface consisting of one or more components of $F \cap B$. If F_0 does not separate the strings of the tangle then each path in B joining the strings must meet F_0 in an even number of points, so F_0 intersects each bubble in an even number of discs, and so the number of curves and arcs cross ∂B_{i+} on each side of

$k \cap \partial B_{i+}$ is even. We claim that in these conditions $F \cap D_+$ consists of pairs of parallel arcs.

To show this, order the arcs according to its distance from k_1, and assume that the first $2n$ are paired and let a be the next one. Let B_i and B_j be two consecutive bubbles crossed by a, so the segments of $k \cap \partial B_{i+}$ and $k \cap \partial B_{j+}$ are on opposite sides of a as in Figure 8a. Since all the curves on one side of a are paired and each side of the bubbles is crossed by an even number of arcs, there must be other arcs a' and a'' crossing B_{i+} and B_{j+} next to a. If a' and a'' are different, then one of them cannot separate k_1 from k_2 (see Figure 8b). If $a' = a''$ then either a and a' run parallel from B_i to B_j or else a' crosses other bubbles between B_i and B_j. If so, let B_l be the bubble crossed by a' immediately after B_j. See Figure 8c. Then $k \cap \partial B_{l+}$ lies between a and a', and so there must be another arc between a and a', and this arc would have to cross B_i or B_j between a and a', and this is impossible. Therefore a' must run parallel to a from the first bubble to the last bubble crossed by a. It remains to show that a' runs parallel to a from the first bubble to the boundary of D_+ and from the last bubble to the boundary of D_+, i.e., that a' does not meet other bubbles in its way to the boundary and that the region between a and a' does not contain other bubbles. As k_1 and k_2 lie outside the region between a and a', this region does not contain any other arc a''. So k^1 and k^2 also lie outside this region, because if k^1 were between a and a' the number of arcs between k_1 and k^1 would be odd, so F_0 would separate these strings of k. Now if there were any segments of $k \cap D_+$ in that region, k would have to enter and leave the region at 2 bubbles crossed by a' on its way to the boundary. But we know that for any two consecutive bubbles crossed by a' the segments of k in their upper hemispheres lie on opposite sides of a', so one of them is in the region between a and a' and so there must be an arc in that region, a contradiction.

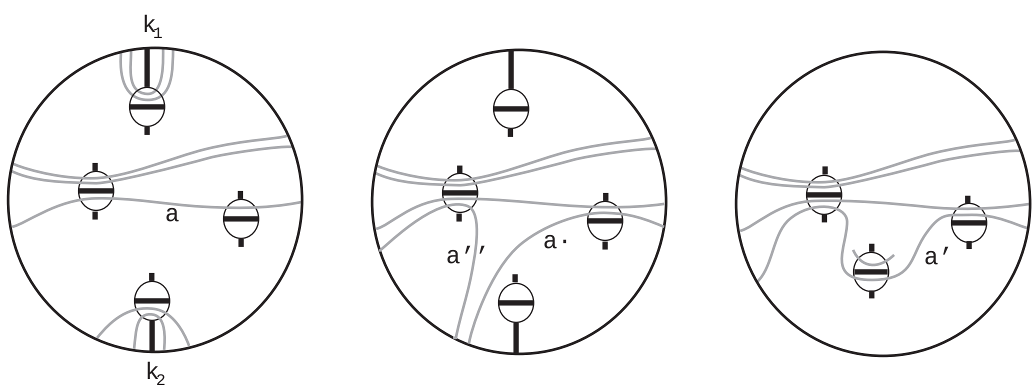

Figure 8

Now observe that each pair of parallel arcs of F_0 in D_+ must be adjacent to a pair of parallel arcs of F_0 in ∂B_+: an arc of F_0 in ∂B_+ cannot go around the endpoints of k^1 or k^2 because F would be meridionally compressible and something analogous holds for the arcs of F_0 in D_-. So the intersection of F_0 with ∂B_+, ∂B_- and with each ∂B_i consists of pairs of parallel curves, and as F_0 is assembled by attaching discs to these parallel curves, F_0 must consist of pairs of parallel surfaces.

Finally, as F is meridionally incompressible, the intersection of F with the 4-punctured sphere $\partial B - k$ consists of curves surrounding 2 punctures, and if there

is more than one curve these are parallel. So if $F \cap B$ has several components, and F_0 consists of any two of them, then any path in B joining the strings of the tangle must intersect F_0 in an even number of points, and this is all that we needed before to show that F_0 consists of parallel surfaces. □

Proof of claim 2. Isotope F to minimize its intersection with ∂B_i. By the previous lemma if $F \cap B_i$ is not empty then it consists of parallel copies of a surface F_0 that separates the strings of the tangle. As k is a knot F cannot separate the strings of $k \cap B$, so there must be an even number of copies of F_0. Now by the previous claim either $F \cap B_i$ or $F \cap S^3 - B_i$ consists of discs, and in the second case F would be the union of the components of $F \cap B_i$ with discs, and since there are at least two such components F would not be connected. □

In [2] Adams *et al.* extended the Meridional Lemma of Menasco to almost alternating knots, i.e. knots that can be obtained by changing one crossing of an alternating knot. The following corollary extends it to knots that can be obtained from an alternating one by mirroring any (2-string) tangle.

Corollary 1. *If a knot k can be decomposed into 2 alternating tangles, then k admits no meridionally incompressible surfaces in its complement.*

Proof. By lemma 1, a meridionally incompressible surface F in the complement of k can be isotoped to meet each of the balls B_1 and B_2 that determine the tangles along an even number of parallel copies of a surface F_i that separates the strings of the tangle.

So $F \cap B_i$ is the boundary of a regular neighborhood N_i of one or more copies of F_i, and N_i is determined by painting the components of $B_i - F$ in a chessboard fashion and choosing those whose color is different from that of the regions that contain the strings of the tangle. So N_1 and N_2 match on $\partial B_1 = \partial B_2$ to form the regular neighborhood of a single surface in S^3, and F is its boundary, so F cannot be connected. □

Corollary 2. *The total genus of a disjoint family of closed, embedded, totally geodesic surfaces in a hyperbolic 3-manifold or link complement is bounded above by:*

- $\frac{3}{2}t$ *where t is the number of tetrahedra in a triangulation.*
- $n - \frac{3}{2}g$ *for manifolds of Heegaard genus g and complexity n.*
- $\frac{3}{2}c - 3$ *for a link with c crossings.*
- $\frac{5}{2}r - 3$ *for a link that admits a projection made of r rational tangles.*
- $\frac{5}{2}r - 3$ *for a prime knot decomposed into alternating tangles, r of them rational.*

Proof. If M is a hyperbolic 3-manifold and F_1,F_2,...,F_k are disjoint totally geodesic surfaces in M, then each F_i is acylindrical and there are no essential annuli in M connecting two F_i's. For, the preimages of the F_i's in the universal covering of M are disjoint totally geodesic planes in $\mathbb{H}^3$, and each preimage of an essential annulus is an infinite strip of bounded height connecting two lines in different planes. These lines lie at a bounded distance from geodesic lines representing the preimages of the boundaries of the annulus, so they determine 2 different points at infinity where the two planes meet, but two disjoint totally geodesic planes in H^3 can only meet at 1 point.

So we may consider the family F_1,F_2,...,F_k as a single disconnected acylindrical surface. The arguments above show the existence of essential annuli for a surface F if the rank of $H_1(F)$ is higher than the number of bad edges, independently of the number of components of F. The bounds arise from a count of the number of bad edges in each case. □

3. Acylindrical surfaces in tunnel number two complements

Let S be a closed surface of genus g standardly embedded in S^3, that is, it bounds a handlebody on each of its sides. A knot K has a (b,g)-presentation if can be isotoped to intersect S transversely in $2b$ points that divide K into $2b$ arcs, so that the b arcs in each side can be isotoped, keeping the endpoints fixed, to disjoint arcs on S. We say that a knot K is a (b,g)-knot if it has a (b,g)-presentation. Consider a product neighborhood $S \times I$ of S. To say that a knot K has a (b,g)-presentation is equivalent to say that K can be isotoped to lie in $S \times I$, so that $K \cap (S \times \{0\})$ and $K \cap (S \times \{1\})$ consist each of b arcs (or b tangent points), and the rest of the knot consist of $2b$ straight arcs in $S \times I$, that is, arcs which intersect each leave $S \times \{t\}$ in the product exactly in one point. It is not difficult to see that if K is a (b,g)-knot, then the tunnel number of K, denoted $tn(K)$, satisfies $tn(K) \leq b+g-1$. In this section we construct $(2,1)$-knots, which are in fact tunnel number 2 knots, which contain an acylindrical surface of genus g.

Let T be a standard torus in S^3, and let $I=[0,1]$. Consider $T \times I \subset S^3$. $T \times \{0\}$ bounds a solid torus R_0, and $T \times \{1\}$ bounds a solid torus R_1, such that $S^3 = R_0 \cup (T \times I) \cup R_1$. Choose $n+1$ distinct points on I, $e_0 = 0,\ e_1, \dots, e_n = 1$, so that $e_i < e_{i+1}$, for all $0 \leq i \leq n-1$. Consider the tori $T \times \{e_i\}$. By a vertical arc in a product $T \times [a,b]$ we mean an embedded arc which intersects every torus $T \times \{x\}$ in the product in at most one point.

Let γ_i be a simple closed essential curve embedded in the product $T \times [e_{i-1}, e_i]$, for $i = 1, \dots, n$, so that it has only one local maximum and one local minimum with respect to the projection to $[e_{i-1}, e_i]$. Let α_i, for $i = 1, \dots, n-1$, be a vertical arc in $T \times [0,1]$, joining the maximum point of γ_i with the minimum of γ_{i+1}. Let Γ be the 1-complex consisting of the union of all the curves γ_i and the arcs α_j. So Γ is a trivalent graph embedded in S^3. Let $R_0' = R_0 \cup (T \times [e_0, e_1])$ and $R_1' = R_1 \cup (T \times [e_{n-1}, e_n])$.

Suppose each curve γ_i satisfies the following:

(1) γ_i is not in a 3-ball contained in $T \times [e_{i-1}, e_i]$, or in R_0' or R_1', that is, it is not a trivial knot in that region.
(2) γ_i is not isotopic in $T \times [e_{i-1}, e_i]$, or in R_0' or R_1', to a knot lying on the torus $T \times \{e_i\}$.
(3) γ_i is not a cable of a knot lying in $T \times [e_{i-1}, e_i]$ or in R_0' or R_1' (it can be proved that this is equivalent to say that γ_i is not isotopic to a cable of a knot lying on the torus $T \times \{e_i\}$.)
(4) There is no annulus B in $T \times \{e_0\}$ so that $B \times [0,1]$ contains Γ. If that happens then each curve γ_i would be contained in a product $B \times [e_{i-1}, e_i]$.
(5) There is no Möbius band in R_0' (R_1') disjoint from γ_1 (γ_n).

It is not difficult to see that there exist plenty of knots satisfying the conditions required for the curves γ_i, say by taking each γ_i to be a $(1,1)$-knot which is not a torus knot nor a satellite knot. For example, each γ_i could be a copy of the figure eight knot, as shown in Figure 9(a) in the case of γ_1, Figure 9(b) for $\gamma_2, \dots, \gamma_{n-1}$,

and Figure 9(c) for γ_n. In the figures the knot is divided in two arcs; the thin arc contains the minimum point of the knot, and the bold arc contains the maximum. When assembled we get the graph Γ, shown for $n = 2$ in Figure 10.

Figure 9

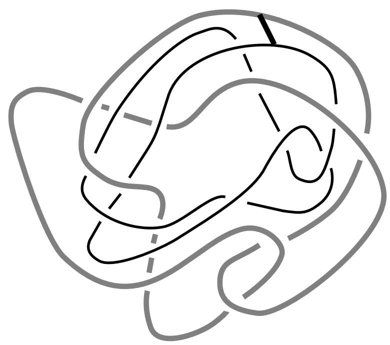

Figure 10

Let $N(\Gamma)$ be a regular neighborhood of Γ. This is a genus n handlebody. We can assume that $N(\Gamma)$ is the union of n solid tori $N(\gamma_i)$, joined by $(n-1)$ 1-handles $N(\alpha_j)$.

Theorem 3. *Let Γ be a graph as above. Then $S = \partial N(\Gamma)$ is incompressible and acylindrical in $S^3 - intN(\Gamma)$. Furthermore, $M - intN(\Gamma)$ is atoroidal.*

Proof. Consider the tori $T \times \{e_i\}$, $1 \leq i \leq n-1$. These tori divide S^3 into $n+1$ regions, where $n-1$ of them are product regions and two of them are solid tori, namely R_0' and R_1'. The torus $T \times \{e_i\}$ intersects Γ in one point, that is, a middle point of α_i, so $T \times \{e_i\} \cap N(\Gamma)$ consists of a disc. Let $T_i = T \times \{e_i\} - intN(\Gamma)$, for $1 \leq i \leq n-1$, this is a once punctured torus.

Suppose D is a compression disc for S, and suppose it intersects transversely the tori T_i. Let β be a simple closed curve of intersection between D and the collection of tori, which is innermost in D. So β bounds a disc $D' \subset D$, which is contained in a product $T \times [e_{i-1}, e_i]$, or in the solid torus R_0' or in R_1'. If β is trivial on T_i, then by cutting D with an innermost disc lying in the disc bounded by β on T_i, we get a compression disc with fewer intersections with the $T_i's$. If β is essential on T_i, then it would be parallel to ∂T_i, or it would be a meridian of T_1 or a longitude of T_{n-1}, but then in any case, one of the curves γ_1 or γ_n will be contained in a 3-ball, which is a contradiction.

So suppose D intersects the $T_i's$ only in arcs. Let β such an arc which is outermost on D, then it cobounds with an arc $\delta \subset \partial D$ a disc D'. We can assume that β is an arc properly embedded in some T_i; if β is parallel to an arc on ∂T_i, then by cutting D with an outermost such arc lying on T_i we get another compression disc with fewer intersections with the $T_i's$, so assume that β is an essential arc on T_i. After isotoping D if necessary, we can assume that the arc δ can be decomposed as $\delta = \delta_1 \cup \delta_2 \cup \delta_3$, where δ_1, δ_3 lie on $\partial N(\alpha_i)$ and δ_2 lie on $\partial N(\gamma_i)$ (if δ were contained in $\partial N(\alpha_i)$, then by isotoping D we would get a compression disc intersecting T_i in a simple closed curve). Let E be a disc contained in $N(\alpha_i)$ so that $\partial E = \delta_1 \cup \delta_4 \cup \delta_3 \cup \delta_5$, where δ_4 lies on T_i and δ_5 lies on $\partial N(\alpha_i)$. So $D' \cup E$ is an annulus, where one boundary component, i.e., $\beta \cup \delta_4$ lies on $T \times \{e_i\}$, and the other, $\delta_2 \cup \delta_5$, lies on $\partial N(\gamma_i)$. If $\delta_2 \cup \delta_5$ is a meridian of γ_i, then necessarily $D \cup E$ is contained in R'_0 (or in R'_1) and $\beta \cup \delta_4$ is a meridian of that solid torus. Then γ_1 (or γ_n) intersects a meridian disc of R'_0 (R'_1) in one point, which implies that it is parallel to a knot lying on the torus $T \times \{e_0\}$ ($T \times \{e_1\}$), which is a contradiction. If $\delta_2 \cup \delta_5$ is a longitudinal curve of γ_i, then this implies that γ_i is parallel to a curve on $T \times \{e_i\}$, a contradiction. If $\delta_2 \cup \delta_5$ goes more than once longitudinally on γ_i, this would only be possible for the curves γ_1 or γ_n, but then one of these curves would be a core of the solid torus R'_0 or R'_1, which is not possible. This completes the proof that S is incompressible in $S^3 - intN(\Gamma)$.

Suppose now that there is an essential annulus A in $S^3 - intN(\Gamma)$. Look at the intersection between A and the punctured tori T_i. Simple closed curves of intersection which are trivial on A, and arcs on A which are parallel to a component of ∂A are eliminated as above. So the intersection consists of a collection of essential arcs on A, or a collection of essential simple closed curves on A.

Suppose first that there are essential arcs of intersection. Let $E \subset A$ be a square determined by the arcs of intersection. So $\partial E = \epsilon_1 \cup \delta_1 \cup \epsilon_2 \cup \delta_2$, where ϵ_1, ϵ_2 are contained in different components of ∂A and δ_1, δ_2 are arcs of intersection of A with the $T_i's$. Take the square at highest level. So δ_1, δ_2 lie on the same level T_i, and possibly $T_i = T_{n-1}$. So we can assume that ϵ_1, ϵ_2 lie on $\partial N(\alpha_i \cup \gamma_{i+1})$.

Case 1: The arcs δ_1, δ_2 are parallel on T_i, that is, they cobound a disc F in T_i.

There are two subcases, depending of the orientation of the arcs δ_1, δ_2. Give an orientation to ∂E. Suppose first that the arcs δ_1, δ_2 have the same orientation on T_i (note that the interior of F may intersect the annulus A, but it is irrelevant in this case). Then $E \cup F$ is a Möbius band, and by pushing it off T_i we get a Möbius band contained in the product $T \times [e_i, e_{i+1}]$ or in R'_1, with its boundary lying on $N(\gamma_i)$. This implies that either γ_i is a trivial knot or that it is a 2-cable of some knot, which is a contradiction.

Suppose the arcs δ_1, δ_2 have opposite orientations in T_i. If the interior of the disc F intersects A, then take another square in A, which determines a disc $F' \subset F$ with interior disjoint from A. We can form two annuli, $E \cup F$ and $(A - E) \cup F$. We will show that at least one of them is an essential annulus. Note that a core of A is homotopic to the product of a core of $E \cup F$ and a core of $(A - E) \cup F$. So if these two curves are homotopically trivial, so is the core of A. So assume one of them is incompressible, say $(A - E) \cup F$. If it is ∂-compressible then it is ∂-parallel, because S is incompressible. Then there is a ∂-compression disc for this annulus intersecting it on $(A - E)$, but this implies that the original annulus A is

also ∂-compressible, a contradiction. So we get a new essential annulus with fewer intersection with the $T_i's$.

Case 2: The arcs δ_1, δ_2 are not parallel on T_i, and the arcs ϵ_1, ϵ_2 are parallel on $\partial N(\Gamma)$.

The arcs ϵ_1, ϵ_2 must have the same orientation on $N(\alpha_i \cup \gamma_i)$, see Figure 11(a). They cobound a disc F on $\partial N(\alpha_i \cup \gamma_i)$ with $\partial F = \epsilon_1 \cup \eta_1 \cup \epsilon_2 \cup \eta_2$, where $\gamma_1, \gamma_2 \subset \partial N(\alpha_i) \cap T_i$. (Note that the disc F may intersect the arc α_{i+1}, or its interior may intersect A, but this is irrelevant in this argument). It follows that $E \cup F$ is a Möbius band whose boundary lies on T_i. This is impossible if the band lies in a product region. If it lies in R'_1, then note that the band is disjoint from the curve γ_n, but this is not possible, by hypothesis.

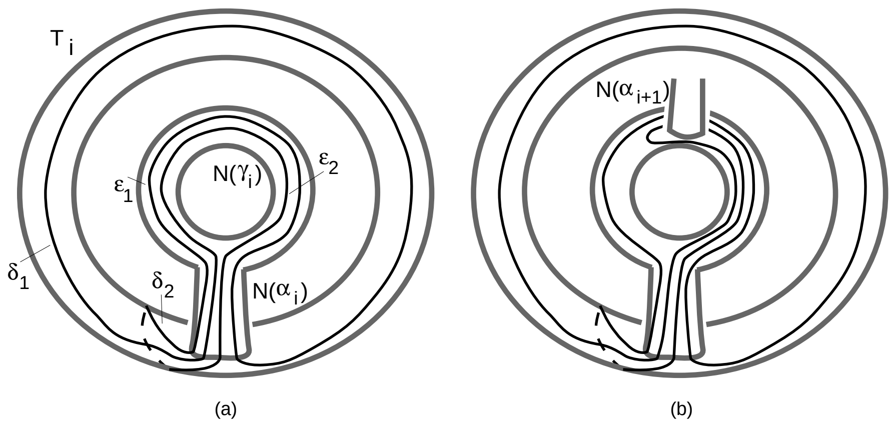

Figure 11

Case 3: The arcs δ_1, δ_2 are not parallel on T_i, and the arcs ϵ_1, ϵ_2 are not parallel on $\partial N(\Gamma)$.

Note that this case is only possible in a product region, see Figure 11(b). Forget about the arc α_{i+1}, that is, consider the square E in the complement of $N(\alpha_i \cup \gamma_i)$. Then, it is not difficult to see that one of the arcs, say ϵ_2 can be slid toward T_i. Then there is a disc, whose boundary consists of two arcs, one lying on T_i and one on $N(\gamma_i)$. By gluing to this disc a disc contained in $N(\alpha_i)$, an annulus between γ_i and $T \times e_i$ is constructed. The only possibility in this case is that the annulus goes once longitudinally on $N(\gamma_i)$, i.e., the curve γ_i is parallel to the torus $T \times e_i$, which is a contradiction.

This completes the proof in the case the annulus A is divided in squares.

Suppose now that the intersection of the annulus A with the tori $T_i's$ consists of simple closed curves which are essential on A. Take an outermost curve, say α. Then α and a component of ∂A cobound an annulus, and the component of ∂A must lie on some γ_i. This again implies that γ_i is parallel to T_i or that γ_1 or γ_n are the core of the solid torus R'_0 or R'_1, a contradiction.

It remains to prove that $S^3 - intN(\Gamma)$ is atoroidal. Suppose Q is an essential torus, then we can assume that it intersects the tori T_i in a collection of simple closed curves which are essential on Q, and divide Q in a collection of annuli. Take one of this annuli, say A, at highest level. If A is in a product region then it must be parallel to some T_i, and then by an isotopy we can remove two curves of

intersection. So A lies on R_1'. As it is an annulus in a solid torus, it must be parallel to the boundary. If γ_n is not in this parallelism region, then an isotopy removes the intersection. If γ_n is the parallelism region, then take the annulus next to A. It must be an annulus between T_{n-1} and T_{n-2}. Continuing in this way, the only possibility is that the whole graph Γ lies inside a solid torus bounded by Q, but this is isotopic to a solid torus of the form $B \times I$, where B is an annulus in $T \times \{e_n\}$. This contradicts the choice of Γ. $\square$

Put now a knot K inside $N(\Gamma)$ in such a way that $K \cap N(\alpha_i)$, for $2 \leq i \leq n-1$, consists of four vertical arcs with a pattern like in Figure 12(c), and $k \cap N(\gamma_i)$ consists of 4 vertical arcs, going from $N(\alpha_i)$ to $N(\alpha_{i+1})$, as in Figure 12(b). Also, $K \cap N(\gamma_1)$ consists of two arcs, each having a single minimum, and $K \cap N(\gamma_n)$ consists of two arcs, each having a single maximum, as in the pattern shown in Figure 12(a). For $n = 3$, a knot K inside $N(\Gamma)$ looks like in Figure 13, where the twist is added to get a knot.

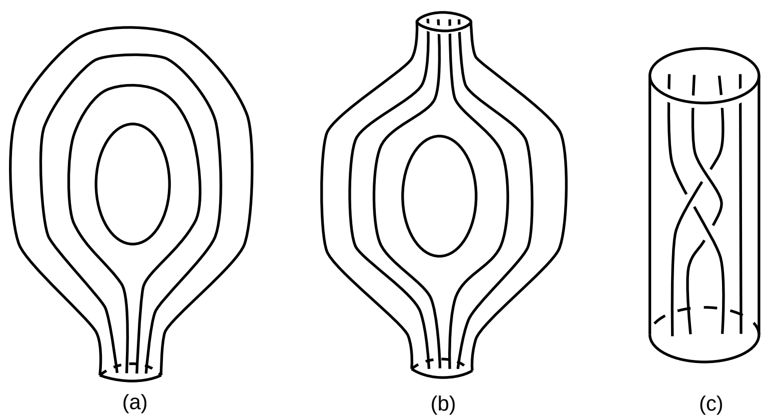

Figure 12

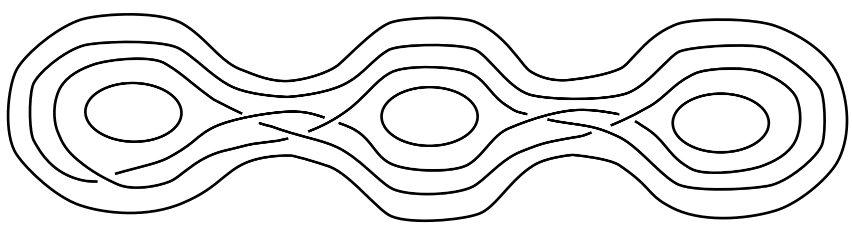

Figure 13

Lemma 2. *$S = \partial N(\Gamma)$ is acylindrical in $N(\Gamma) - K$. Furthermore, $N(\Gamma) - K$ is atoroidal.*

Proof. The proof is also an innermost disc/outermost arc argument. It is practically the same as in Lemma 2.3 of [3]. $\square$

Theorem 4. *Let K and S as constructed above. K is a hyperbolic $(2, 1)$-knot, tunnel number 2 knot, and S is an acylindrical surface of genus g in the complement of K.*

Proof. Note that by construction K is a $(2,1)$-knot, for it lies in $T \times I$, and it has in there exactly two maxima and two minima with respect to the projection to the factor I. It follows from Theorem 3 and lemma 2 that S is an acylindrical surface. K is a hyperbolic knot because the complement of the surface is atoroidal and acylindrical. Finally note that the knot K has tunnel number 2; it cannot have tunnel number one, for it contains an acylindrical separating surface [19]. □

Corollary 3. *Given any integer $g \geq 2$, there exist infinitely many hyperbolic 3-manifolds of Heegaard genus 3 which contain an acylindrical surface of genus g.*

Proof. For each g choose a knot K as above. Do Dehn surgery on K with slope λ, such that $\Delta(\mu, \lambda) \geq 3$, where μ is a meridian of K. It follows that S remains incompressible [22], acylindrical [11], and that $M(\alpha)$ is irreducible and atoroidal [9] [10]. Then by Thurston Geometrization Theorem, $M(\alpha)$ is hyperbolic, for it is Haken and atoroidal. K has tunnel number two, which implies that $M(\alpha)$ has Heegaard genus at most 3, but it cannot have Heegaard genus 2, for it contains a separating acylindrical surface [19]. □

4. Quasi-Fuchsian surfaces of arbitrarily high genus

Let M be an irreducible orientable 3-manifold. Let K be a knot in M. Let $\mathcal{B}$ be a branched surface in M disjoint from K. (see [7] [20] for definitions and facts about branched surfaces). Denote by N a fibered regular neighborhood of $\mathcal{B}$, by $\partial_h N$ the horizontal boundary of N, and by $\partial_v N$ the vertical boundary of N, as usual.

We say that a branched surface $\mathcal{B}$ is incompressible in $M - K$ if it satisfies:

(1) $\mathcal{B}$ has no discs of contact or half discs of contact.
(2) $\partial_h N$ is incompressible and ∂-incompressible in $(M - K) - intN$.
(3) There are no monogons in $(M - K) - intN$.
We further say that $\mathcal{B}$ is meridionally incompressible if:
(4) $\partial_h N$ is meridionally incompressible, that is , there is no disc D in M, with $D \cap N = \partial D \subset \partial_h N$, so that D intersects K transversely in one point.
We further say that K is not parallel to $\mathcal{B}$ if:
(5) K is not parallel to $\partial_h N$, that is, there is no an annulus A in M, with $\partial A = A_0 \cup A_1$, so that $A_0 = K$, and $A \cap N = A_1 \subset \partial_h N$

Theorem 5. *Let M, $\mathcal{B}$, K as above, with $\mathcal{B}$ incompressible.*

(1) *Suppose $\mathcal{B}$ is meridionally incompressible. Then a surface carried with positive weights by $\mathcal{B}$ is meridionally incompressible.*
(2) *If K is not parallel to $\mathcal{B}$, then K is not parallel to any surface carried with positive weights by $\mathcal{B}$.*

Then if $\mathcal{B}$ is meridionally incompressible and K is not parallel to it, any surface carried by $\mathcal{B}$ with positive weights is quasi-Fuchsian.

Proof. It is essentially the same proof as in Theorem 2 in [7], with the obvious modifications. □

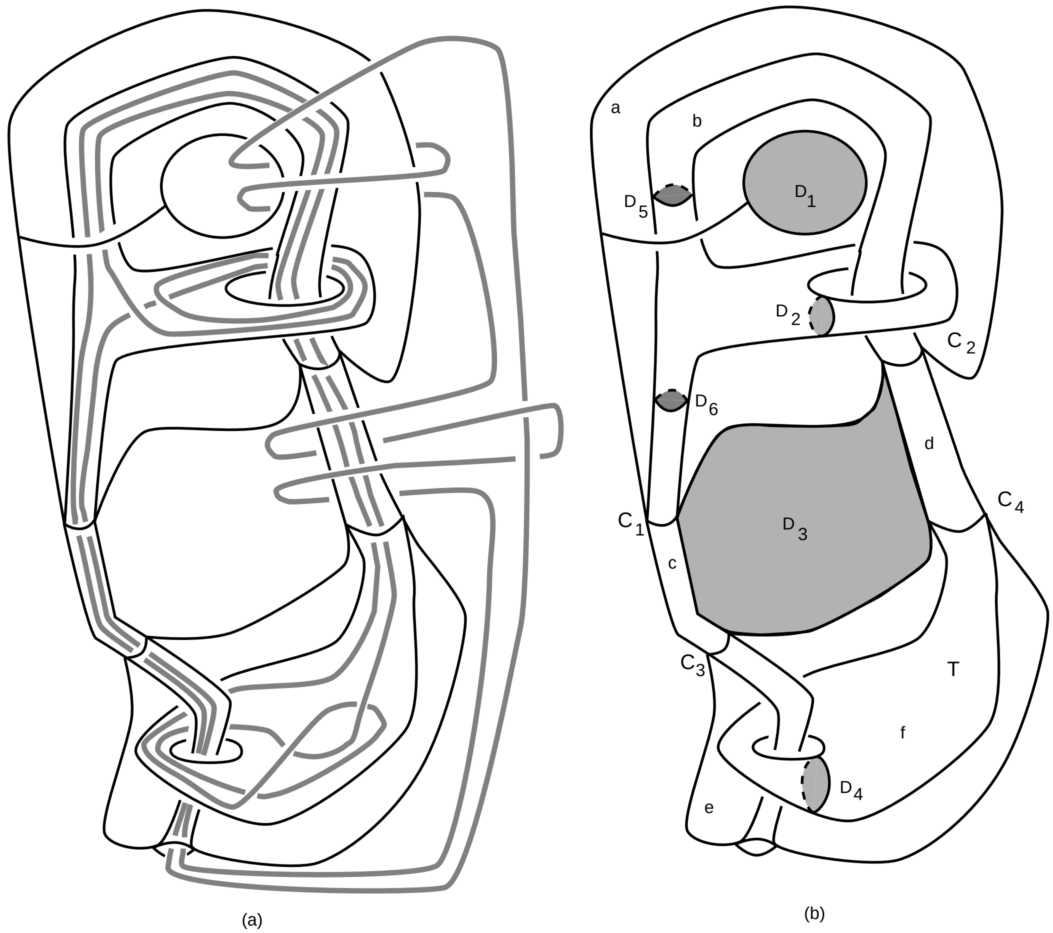

Figure 14

Consider the knot K and branched surface $\mathcal{B}$ shown in Figure 14(a). Note that $\mathcal{B}$ has 4 singular curves, denoted $C_1,\ C_2,\ C_3,\ C_4$, as in Figure 14(b). Note that K is a 3-bridge knot. The knot K is just one in a collection of knots, to get more just make the knot to intersect several times the discs D_1, D_2, D_3, D_4 shown in Figure 14(b). But suppose that K intersects transversely the discs D_1, D_2, D_3, D_4 in at least 2 points, that is, the minimal intersection number of the knots with the discs, when isotoping the knot in the complement of $\mathcal{B}$ is 2. Note that K intersects the discs D_5 and D_6 in exactly 2 points, because it is a 3-bridge knot. Suppose also that the arc of the knot lying in the solid torus T (shown in Figure 14(b)), is not parallel to ∂T; it is possible to do that, an explicit example is in Figure 14(a).

The nonsingular part of $\mathcal{B}$ has six components, whose weights (a, b, c, d, e, f) are shown in Figure 14(b). Note that if we give the weights $(1, 2n-1, 2n, 2n-2, n, n-2)$, for $n \geq 3$, then this is a collection of positive weights, which is consistent, and determines a connected surface of genus $3n$.

If a knot K is not hyperbolic then it is either a torus knot or a satellite knot. Remember that by the classical work of Schubert, a satellite 3-bridge knot must be the connected sum of 2 two-bridge knots. It is known that two-bridge knots do

not contain any essential closed surface [13], and from this it follows that the only essential surfaces in the connected sum of 2 two-bridge knots are the swallow-follow tori. Also, torus knots do not contain closed essential surfaces. This implies that a 3-bridge knot which contains an essential surface of genus greater than 1 must be hyperbolic.

Theorem 6. *The surface $\mathcal{B}$ is meridionally incompressible and K is not parallel to it. So K is a hyperbolic 3-bridge knot which contains quasi-Fuchsian surfaces of arbitrarily high genus.*

Sketch of proof. Let N be a fibered neighborhood of $\mathcal{B}$. Note that $S^3 - N$ has 3 components, denoted by N_1, N_2, N_3, where say N_3 is the region that contains the knot, N_1 is the upper region, and N_2 the lower region.

Suppose that the part of $\partial_h N$ contained in N_3 is compressible or meridionally compressible, and let E be a compression or meridian compression disc. Look at the intersections between E and the discs D_1, D_3, D_5, D_6. Let γ be a simple closed curve of intersection which is innermost on E, so γ bounds a disc $E' \subset E$; suppose first that E' is disjoint from K. The curve γ also bounds a disc D' in some D_i. Suppose D' intersects K. If D' is part of D_1 or D_3, then K intersects the sphere $E' \cup D'$ several times always in the same direction, which is impossible. If D' is part of D_5 or D_6 then it must intersect K in two points, and then there is an arc of K contained in the 3-ball bounded by $E' \cup D'$. But this implies that K can be made disjoint from D_2 or D_4, or from D_3 or D_1, which is impossible by hypothesis. So D' must be disjoint from K, and then an isotopy reduces the number of intersections between E and the D_i. If E' intersects K once, then by a similar argument, D' intersects K also in a point, and then by an isotopy, we get a new compression disc with fewer intersections with the D_i. Suppose then that the intersection between E and the D_i consists only of arcs. Let γ be an arc of intersection which is outermost on E, and which bounds a disc E' disjoint from K. The arc γ also bounds a disc D' on some D_i. If K is disjoint from D', then by cutting E with an outermost disc lying on D' we get a new compression disc with fewer intersections with the D_i. If K intersects D' in one point, then it is not difficult to see that K must intersect in one point one of D_2, D_4 or D_1, which is a contradiction. So if there is such a disc E, it must be disjoint from the D_i, and by inspection it is not difficult to check that such disc does not exist. The part of $\partial_v N$ contained in N_3 consists of one annulus, corresponding to the curve C_2. Again an innermost disc/outermost arc argument shows that there is no monogon.

The part of $\partial_h N$ contained in N_1 consists of a twice punctured genus two surface; it is not difficult to check that it is incompressible. The part of $\partial_v N$ contained in N_1 consists of an annulus, corresponding to the curve C_1; it is also not difficult to check that there is no monogon. Similarly, the part of $\partial_h N$ contained in N_2 consists of a three punctured sphere and an once punctured torus, and $\partial_v N$ consists of two annuli, corresponding to the curves C_3 and C_4; again it is not difficult to check that these are incompressible and that there is no monogon.

To see that K is not parallel to $\mathcal{B}$, suppose there is an annulus A, with one boundary being K and the other on $\mathcal{B}$. Again look at the intersections between A and the discs D_i, and get that the arc of the knot that lies in the solid torus T must be parallel to ∂T, but this is not possible by the choice of such an arc. $\square$

The explicit knot shown in Figure 14(a) has more interesting properties, it is a ribbon knot and it has unknotting number one, where a crossing change is located in the arc contained in the solid torus T.

Corollary 4. *There exist hyperbolic genus 3 closed 3-manifolds, in fact homology spheres, which contain incompressible surfaces of arbitrarily high genus, so contain infinitely many incompressible surfaces.*

Proof. Let K be a knot as in Theorem 6. Let $K(r)$ be the manifold obtained by performing Dehn surgery on K with slope r. If $\Delta(r,\mu) > 1$, where μ denotes a meridian of K, then $K(r)$ is irreducible by [9], and $\mathcal{B}$ remains incompressible in $K(r)$ by [22], for K is not parallel to $\mathcal{B}$. If $\Delta(r,\mu) > 2$, then $K(r)$ is atoroidal by [10]. So if $\Delta(r,\mu) > 2$, $K(r)$ is an atoroidal Haken manifold, hence it is hyperbolic. K is a tunnel number 2 knot, hence each $K(r)$ has Heegaard genus ≤ 3. Finally note that among the $K(r)$ many are homology spheres. □

Corollary 5. *There exist genus 2 closed 3-manifolds which contain incompressible surfaces of arbitrarily high genus, so they contain infinitely many incompressible surfaces.*

Proof. Let K be a knot as in Theorem 6. Let $\Sigma(K)$ denote the double cover of S^3 branched along K. As K is a 3-bridge knot, $\Sigma(K)$ has Heegaard genus 2. If S is a surface carried by $\mathcal{B}$ with positive weights, then as it is meridionally incompressible, it lifts in $\Sigma(K)$ to a (possible disconnected) incompressible surface [8]. □

Remark 1. *It should be possible to say that the manifolds obtained in this corollary are hyperbolic; this will be the case if it is shown that the knots K do not admit a tangle decomposing sphere.*

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